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NOTE

A Note on Magnetic Monopoles and the One-Dimensional MHD Riemann Problem

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1. INTRODUCTION

The evolution in time of a magnetic field **B** is determined by an electric field **E** through the induction equation [4, 8, 12, 14]

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{1}$$

one of Maxwell's equations. The magnetic field must also satisfy $\nabla \cdot \mathbf{B} = 0$. This constraint expresses the absence of magnetic monopoles, which have never been observed experimentally. Since (1) implies $\partial_t(\nabla \cdot \mathbf{B}) = 0$ this constraint is often treated as an initial condition, which will be preserved under subsequent evolution.

The induction equation (1) is combined with the equations of gas dynamics to describe the behaviour of compressible electrically conducting fluids subject to magnetic fields. For non-relativistic fluids, where Maxwell's displacement current may be neglected, the combined system is referred to as the magnetohydrodynamic (MHD) equations. The compressible ideal (inviscid and perfectly conducting) MHD equations may be written as a hyperbolic system of conservation laws in the form [7, 9, 11]

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \mathbf{B} \\ U \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathsf{I} - \mathbf{B}\mathbf{B} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \\ \left(U + p + \frac{1}{2}B^2\right)\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B} \end{bmatrix} = 0, \tag{2}$$

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using a standard notation in which $\mathbf{v}$ is the fluid velocity, ρ the density, and p the pressure. The electric field $\mathbf{E}$ has been eliminated using the perfectly conducting condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ appropriate for highly conductive media such as astrophysical plasmas. The symbol I denotes the 3×3 identity tensor with components δ_{ij}. A factor of $1/\sqrt{\mu_0}$ has been absorbed into the definition of $\mathbf{B}$. The total energy density U is given by

$$U = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 + \frac{1}{2}B^2, \tag{3}$$

for a perfect gas with ratio of specific heats γ. The three terms comprise the internal, kinetic, and magnetic energy, respectively.

In recent years Godunov-type upwind methods have become popular for solving hyperbolic systems such as (2), especially when shocks are likely to form [3, 7, 10]. These finite volume methods often compute upwind fluxes by solving the one-dimensional system for initial data comprising uniform left and right states separated by a single discontinuity at a computational cell boundary, i.e., by solving the one-dimensional Riemann problem [3, 7].

2. THE MHD RIEMANN PROBLEM

Unfortunately the one-dimensional MHD equations are degenerate. If the variables in (2) are functions of one coordinate n only, the evolution equation for B_n becomes simply $\partial_t B_n = 0$. In a sense, this is consistent with the one-dimensional form of the constraint $\nabla \cdot \mathbf{B} = 0$, which simplifies to $\partial_n B_n = 0$. Thus the magnetic field B_n normal to the discontinuity should be constant in the initial conditions and will then remain constant.

In a finite volume discretisation, $\nabla \cdot \mathbf{B} = 0$ implies that the signed sum of the jumps in the normal component of $\mathbf{B}$ across a cell's boundary should vanish. Thus the initial data for the one-dimensional Riemann problems, if used to compute fluxes in a multidimensional scheme, will generally contain jumps in B_n comparable to the jumps in the tangential components $\mathbf{n} \times \mathbf{B}$.

One approach is to use the solution of the reduced seven-wave Riemann problem [13], using some average value for B_n on both sides, to update the seven variables other than B_n, followed by a separate step which updates B_n so as to preserve $\nabla \cdot \mathbf{B} = 0$ [2, 16]. An alternative approach, pioneered by Powell *et al.* [9, 10] (see also [11]), adds terms proportional to $\nabla \cdot \mathbf{B}$ to the system (2) to make the one-dimensional Riemann problem non-degenerate. It is worth emphasising that although Powell's approach maintains $\nabla \cdot \mathbf{B} \approx 0$ to truncation error in a multidimensional sense, in the one-dimensional Riemann problems $\nabla \cdot \mathbf{B}$ is comparable to $\nabla \times \mathbf{B}$. Numerical experiments comparing various schemes have been performed recently by Tóth [15].

3. POWELL'S EIGHT-WAVE MODIFICATION

Powell *et al.* [9, 10] (see also [11]) proposed solving the modified system

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \mathbf{B} \\ U \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathsf{I} - \mathbf{B}\mathbf{B} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \\ \left(U + p + \frac{1}{2}B^2\right)\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B} \end{bmatrix} = -\nabla \cdot \mathbf{B} \begin{bmatrix} 0 \\ \mathbf{B} \\ \mathbf{v} \\ \mathbf{v}\cdot\mathbf{B} \end{bmatrix}, \tag{4}$$

in which source terms proportional to $\nabla \cdot \mathbf{B}$ have been added to the momentum and induction and energy equations. This system was constructed by modifying the coefficient matrix in the linearised Riemann problem to include an eighth wave corresponding to passive advection of jumps in B_n with the fluid speed v_n. This is the only possibility that leaves the system invariant under the Galilean transformations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}_0 t$ and $\mathbf{v} \mapsto \mathbf{v} + \mathbf{v}_0$, while $\mathbf{B}$ and t remain unchanged.

Janhunen [5] found that the solution of the Riemann problem for Powell's system (4) for left and right states with positive fluid pressures may contain an unphysical intermediate state with negative fluid pressure. This so-called lack of positivity [3] is a particular problem for astrophysical applications, where the contribution to the total energy from the fluid pressure is often small compared with the magnetic and possibly the kinetic energy. Thus computing the fluid pressure p from the conserved quantities ρ, $\rho\mathbf{v}$, $\mathbf{B}$, and U often involves the difference between two nearly equal terms. Janhunen [5] found that positivity, as well as local energy and momentum conservation, could be regained by discarding the source terms in the energy and momentum equations, so that (4) becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \mathbf{B} \\ U \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathsf{I} - \mathbf{B}\mathbf{B} \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \\ \left(U + p + \frac{1}{2}B^2\right)\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B} \end{bmatrix} = -\nabla \cdot \mathbf{B} \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \\ 0 \end{bmatrix}. \tag{5}$$

4. DERIVATION FROM RELATIVISTIC ENERGY-MOMENTUM CONSERVATION

We now give a systematic derivation of the system (5). In special relativity, energy and momentum conservation are expressed compactly as a single conservation law for a four-dimensional stress-energy tensor [6, 8, 12]

$$\partial_\beta \left(T_{\mathrm{FL}}^{\alpha\beta} + T_{\mathrm{EM}}^{\alpha\beta} \right) = 0, \tag{6}$$

where $T_{\mathrm{FL}}^{\alpha\beta}$ and $T_{\mathrm{EM}}^{\alpha\beta}$ are the separate fluid and electromagnetic contributions to the stress–energy tensor. Our notation follows Misner *et al.* [8] except we retain explicit factors of c, the speed of light. Greek indices range over 0, 1, 2, 3, with 0 being the time-like component, and Latin indices range over the space-like components 1, 2, 3. A coordinate vector is thus $x^\alpha = (ct, \mathbf{x})$, while the metric is $\mathbf{G} = \mathrm{diag}(-1, 1, 1, 1)$.

The four-dimensional stress–energy tensor for a relativistic ideal fluid is [6, 8, 12]

$$T_{\mathrm{FL}}^{\alpha\beta} = (p + e)u^\alpha u^\beta + p g^{\alpha\beta}, \tag{7}$$

where p is the pressure and e the relativistic energy density. In the non-relativistic limit, $|\mathbf{v}| \ll c$, the various components of $T_{\mathrm{FL}}^{\alpha\beta}$ become [6]

$$T_{\mathrm{FL}}^{00} = \rho c^2 + \epsilon + \frac{1}{2}\rho v^2, \quad T_{\mathrm{FL}}^{0i} = T_{\mathrm{FL}}^{i0} = \rho c\mathbf{v} + \frac{\mathbf{v}}{c}\left(p + \epsilon + \frac{1}{2}\rho v^2\right), \quad T_{\mathrm{FL}}^{ij} = p\delta_{ij} + \rho v_i v_j, \tag{8}$$

where the non-relativistic internal energy is $\epsilon = \frac{p}{\gamma - 1}$ for a perfect gas. Unlike Misner *et al.* [8], we use ρ for the rest mass density, rather than the combined mass–energy density, in agreement with standard fluid dynamical usage. The electromagnetic stress–energy tensor has components [4, 8, 12]

$$T^{00}_{\mathrm{EM}} = \frac{1}{2}\left(B^2 + \frac{1}{c^2}E^2\right), \quad T^{0i}_{\mathrm{EM}} = T^{i0}_{\mathrm{EM}} = \frac{1}{c}(\mathbf{E}\times\mathbf{B})_i, \quad T^{ij}_{\mathrm{FL}} = M_{ij}. \tag{9}$$

We recognise T^{00}_{EM} as the electromagnetic energy density, and $T^{0i}_{\mathrm{EM}} = T^{i0}_{\mathrm{EM}}$ as the Poynting flux. The remaining components T^{ij}_{EM} comprise the three-dimensional Maxwell stress $\mathbf{M}$,

$$T^{ij}_{\mathrm{EM}} = M_{ij} = \frac{1}{2}\left(B^2 + \frac{1}{c^2}E^2\right)\delta_{ij} - \left(B_i B_j + \frac{1}{c^2}E_i E_j\right). \tag{10}$$

The components of (9) are unchanged by “duality rotations” [4, 5, 8] of the electromagnetic field, under which $\mathbf{E}$ and $\mathbf{B}$ transform to

$$\mathbf{E}' = \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \quad \mathbf{B}' = \mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha, \tag{11}$$

where α is a real parameter, not to be confused with a component index. Rindler [12] notes that it is possible to construct a relativistic field theory in which the electric and magnetic fields have completely interchangeable status, so $\nabla\cdot\mathbf{E} = \rho_{\mathrm{e}}$ and $\nabla\cdot\mathbf{B} = \rho_{\mathrm{m}} \neq 0$, and which contains conventional electromagnetism ($\nabla\cdot\mathbf{B} = 0$) as a special case. This theory, while permitting both electric and magnetic charges, requires no changes to the stress–energy tensor in (9), which is already symmetric between $\mathbf{E}$ and $\mathbf{B}$.

The $\alpha = 0$ component of (6) corresponds to energy conservation, and $\alpha = 1, 2, 3$ to the three components of momentum conservation. The four-dimensional derivative is $\partial_\beta = \partial/\partial x^\beta = (\frac{1}{c}\partial_t, \nabla)$. From the $\alpha = 0$ component of (6) we obtain the energy equation

$$\frac{1}{c}\frac{\partial}{\partial t}(\rho c^2 + U) + \nabla\cdot\left(\rho c\mathbf{v} + \frac{\mathbf{v}}{c}\left(\frac{1}{2}\rho v^2 + p + \epsilon\right) + \frac{1}{c}\mathbf{E}\times\mathbf{B}\right) = 0. \tag{12}$$

The leading order terms, which reflect the contribution of the rest mass to the relativistic energy, exactly cancel by virtue of the continuity equation $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$, leaving the non-relativistic energy equation [6]. The Poynting flux simplifies using $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$ to

$$\frac{1}{c}\mathbf{E}\times\mathbf{B} = \frac{1}{c}(-\mathbf{v}\times\mathbf{B})\times\mathbf{B} = \frac{1}{c}(\mathbf{v}B^2 - \mathbf{v}\cdot\mathbf{B}\mathbf{B}), \tag{13}$$

as in (5). From the $\alpha = 1, 2, 3$ components of (6) we obtain the momentum equation

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\rho c v_i + \frac{v_i}{c^2}\left(p + \epsilon + \frac{1}{2}\rho v^2\right) + \frac{1}{c}(\mathbf{E}\times\mathbf{B})_i\right) + \frac{\partial}{\partial x_j}(p\delta_{ij} + \rho v_i v_j + M_{ij}) = 0. \tag{14}$$

The fluid energy flux and the electromagnetic Poynting flux contribute to the momentum density in (14), as well as to the energy flux in (12). This is due to the symmetry of the stress–energy tensors, or that relativity associates momentum with the motion of energy as well as matter [4, 8, 12]. However, these terms in (14) are $\mathcal{O}(v^2/c^2)$ smaller than the expected terms, those which give the non-relativistic momentum equation as it appears in (2) and (5). The electric field’s contributions to the energy and Maxwell stress may also be neglected, since they are $\mathcal{O}(v^2/c^2)$ smaller than the magnetic contributions in the non-relativistic limit. For instance, the three-dimensional Maxwell stress simplifies to just

$$M_{ij} = \frac{1}{2}B^2\delta_{ij} - B_i B_j, \tag{15}$$

and (12) and (14) take the forms in which they appear in (2) and (5). This derivation requires no assumptions about the presence or absence of magnetic monopoles.

More generally, the electromagnetic part of (14) already contains the terms that Janhunen added to the usual Lorentz force to make it invariant under duality rotations (11), by which he obtained the "generalised Lorentz force" [5],

$$\mathbf{f} = -\nabla\cdot\mathsf{M} - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\mathbf{E}\times\mathbf{B}\right) = \frac{1}{c^2}(\nabla\cdot\mathbf{E})\mathbf{E} - \frac{1}{c^2}\mathbf{J}_{\mathrm{m}}\times\mathbf{E} + \mathbf{J}_{\mathrm{e}}\times\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{B}, \tag{16}$$

in combination with the two "generalised Maxwell equations" [4, 5, 14]

$$-\mathbf{J}_{\mathrm{m}} = \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}, \quad \mathbf{J}_{\mathrm{e}} = \nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{17}$$

The Lorentz force $\mathbf{f}$ felt by a fluid is not simply $-\nabla\cdot\mathsf{M}$ because the electromagnetic field itself contains momentum, the Poynting flux. Some of the stress exerted by the electromagnetic field goes into changing the momentum of the field itself, as expressed by the $\partial_t(\frac{1}{c^2}\mathbf{E}\times\mathbf{B})$ term in (16), instead of changing the momentum of the fluid. This difference becomes negligible in the non-relativistic limit.

The $(\nabla\cdot\mathbf{B})\mathbf{B}$ term in (16) is usually discarded, on the assumption that $\nabla\cdot\mathbf{B} = 0$ in reality, leading to the usual expression $\mathbf{J}_{\mathrm{e}}\times\mathbf{B}$ for the Lorentz force exerted by a magnetic field on an electrically conducting fluid. However, the consistent expression for the Lorentz force in the presence of magnetic monopoles remains $\mathbf{f}$ as in (16), which no longer coincides with $\mathbf{J}_{\mathrm{e}}\times\mathbf{B}$. This is why Powell's system (4) fails to conserve momentum and energy.

5. MODIFIED INDUCTION EQUATION

In relativistic electromagnetic theory, the two homogeneous Maxwell equations, $\nabla\cdot\mathbf{B} = 0$ and $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$, are components of the single equation [8, 12]

$$\partial_\alpha G^{\alpha\beta} = 0, \tag{18}$$

where the four-dimensional tensor $G^{\alpha\beta}$ has components

$$G^{\alpha\beta} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix}, \tag{19}$$

in terms of the three-dimensional electric and magnetic fields in a given frame.

In the presence of magnetic monopoles (18) generalises to

$$\partial_\alpha G^{\alpha\beta} = \rho_{\mathrm{m}}^{\beta}, \tag{20}$$

where $\rho_{\mathrm{m}}^{\beta}$ must be a four-vector in order to make the equation invariant under Lorentz transformations, i.e., transformations of the form $(ct, \mathbf{x}) \mapsto (ct', \mathbf{x}')$, where $\mathbf{x}' = \Gamma(\mathbf{x} + \mathbf{v}_0 t)$ and $t' = \Gamma(t + \mathbf{x} \cdot \mathbf{v}_0/c^2)$, and similarly for other four-dimensional objects. Note that $\Gamma = (1 - v_0^2/c^2)^{-1/2} = 1 + \mathcal{O}(v_0^2/c^2)$, and $t' = t + \mathcal{O}(v_0 x/c^2)$, so Lorentz transformations reduce to Galilean transformations in the non-relativistic limit. By analogy with the relativistic equation for baryon conservation (i.e., continuity of mass) [6], the simplest choice is $\rho_{\mathrm{m}}^{\beta} = (\rho_{\mathrm{m}}/c)u^{\beta}$, where $u^{\beta} = \Gamma(c, \mathbf{v})$ is the single fluid four-velocity. The scalar ρ_{m} is the density of monopoles in the local fluid rest frame, where $u^{\beta} = (c, \mathbf{0})$. In a general frame, Lorentz-invariance forces ρ_{m}^{i} to be nonzero. The components of (20) then become

$$\nabla \cdot \mathbf{B} = \Gamma \rho_{\mathrm{m}}, \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = -\Gamma \rho_{\mathrm{m}} \mathbf{v}, \tag{21}$$

the latter of which reduces to the Galilean-invariant, modified induction equation of (4) and (5) in the perfectly conducting and non-relativistic ($\Gamma \to 1$) limits.

6. CONCLUSION

Relativistic energy–momentum conservation, Eqs. (6), (7), and (9), is already invariant under duality rotations and requires no changes to accommodate the possibility of magnetic monopoles ($\nabla \cdot \mathbf{B} \neq 0$). In the non-relativistic limit we derive equations which coincide with those previously proposed by Janhunen [5] and differ from those proposed by Powell [9, 10] in that they retain local conservation of energy and momentum in the presence of monopoles. Our derivation also leads directly to a conservative form of the equations, whereas Janhunen's [5] proceeded via primitive variables.

The possibility of magnetic monopoles, $\nabla \cdot \mathbf{B} = \rho_{\mathrm{m}} \neq 0$, requires a source term proportional to $\nabla \cdot \mathbf{B}$ in the induction equation to preserve Lorentz invariance of the combined system. If magnetic monopoles are treated as particles, the simplest approach, Lorentz invariance leads to the modified induction equation first proposed by Powell [9] and adopted unchanged by Janhunen [5].

The Lorentz force consistent with momentum conservation, $\mathbf{J}_{\mathrm{e}} \times \mathbf{B} + \mathbf{B}\nabla \cdot \mathbf{B}$, is not perpendicular to $\mathbf{B}$ unless $\nabla \cdot \mathbf{B} = 0$. This leads to the phenomenon observed by Brackbill and Barnes [1] in which a supposed steady state is "polluted" by magnetic monopoles accelerating along magnetic field lines. In fact, Tóth [15] recently showed that the Lorentz force computed by a momentum conserving scheme cannot be perpendicular to $\mathbf{B}$, even if $\nabla \cdot \mathbf{B} = 0$ in some discrete sense. Tóth also found that Powell's formulation computed incorrect propagation speeds for strong shocks which were inclined to the computational grid. This is typical of non-conservative formulations [7, 15], and it remains to be seen whether errors in the shock speed are alleviated by restoring just momentum and energy conservation. A modification of these equations, retaining the exact fluid stress from (7) instead of the approximation in (8), may also prove useful for relativistic magnetohydrodynamics.

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